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Linear Stability of Supersonic Cone Boundary Layers

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The effect of the variable surface geometry of a cone on the linear stability of a supersonic boundary layer flowing over it is investigated subject to different quasiparallel flow approximations. It is shown that, if a suitable set of disturbance state variables is chosen for the normal mode analysis, these effects can accurately be accounted for. In fact, a planar coordinate system can be used for the stability analysis of the cone boundary-layer profiles and a simple "correction" can subsequently be applied to obtain an accurate approximation to the spatial growth rates.

Nomenclature

a = speed of sound (nondimensionalized by V_∞^*)
c_p = specific heat at constant pressure of the gas per unit mass (nondimensionalized by $\mathcal{R}^*$)
c_{ph} = phase speed of the disturbances (nondimensionalized by V_∞^*)
E_t = total energy of the mixture per unit volume (nondimensionalized by $\rho_\infty^* \mathcal{R}^* T_\infty^*$)
e_{ij} = strain rate tensor (nondimensionalized by $V_\infty^*/L_{\text{ref}}$)
h_1 = $1 + \kappa y$
h_2 = 1
h_3 = $(r + y \cos \epsilon)^m$ (nondimensionalized by L_{ref}^m)
k = thermal conductivity (nondimensionalized by $\mu_\infty^* \mathcal{R}^*$)
L_{ref} = reference length (dimensional); $x^*/\sqrt{Re_x}$
M = Mach number (ratio of flow speed to corresponding sound speed)
m = 0 for two-dimensional flow; 1 for axisymmetric flow
p = static pressure (nondimensionalized by $\rho_\infty^* \mathcal{R}^* T_\infty^*$)
Q = $h_1 h_3 U$
q = heat flux (nondimensionalized by $\mu_\infty^* \mathcal{R}^* T_\infty^*/L_{\text{ref}}$)
r = distance from the body axis to the surface (nondimensionalized by L_{ref}); see Fig. 1
$\mathcal{R}^*$ = gas constant
Re_L = Reynolds number based on the length scale L_{ref}, $\rho_\infty^* V_\infty^* L_{\text{ref}}/\mu_\infty^*$
Re_x = Reynolds number based on the length scale x^*, $\rho_\infty^* V_\infty^* x^*/\mu_\infty^*$
S_{ij} = stress tensor (nondimensionalized by $\mu_\infty^* V_\infty^*/L_{\text{ref}}$)
T = temperature (nondimensionalized by T_∞^*)
t = time (nondimensionalized by $L_{\text{ref}}/V_\infty^*$)
U = vector of conservative dependent (state) variables
u = mass averaged tangential velocity component (along x coordinate lines; nondimensionalized by V_∞^*)
V = mass averaged velocity vector (nondimensionalized by V_∞^*)
v = mass averaged body-normal velocity component (nondimensionalized by V_∞^*)
w = spanwise (two-dimensional) or circumferential (axisymmetric) velocity component (nondimensionalized by V_∞^*)
x = arc length measured along the coordinate tangent to the body surface and in a plane perpendicular to the leading edge (two dimensional) or containing the body axis (axisymmetric) (nondimensionalized by L_{ref}); see Fig. 1
y = distance measured along the coordinate normal to body surface (nondimensionalized by L_{ref}); see Fig. 1
z = distance measured parallel to the leading edge (two dimensional; nondimensionalized by L_{ref}) or angle measured circumferentially around the body axis (axisymmetric; dimensionless); see Fig. 1
α = (complex) wave number in streamwise direction
β = (real) wave number in the spanwise or circumferential direction
δ^* = boundary-layer length scale; $x^*/\sqrt{Re_x}$
Δ^* = shock standoff distance; see Fig. 1
ϵ = angle between body surface and body axis; see Fig. 1
γ = ratio of specific heats
κ = $-(d\epsilon/dx)$(nondimensionalized by L_{ref})
μ = dynamic viscosity (nondimensionalized by the freestream value)
π = ratio of the circumference of a circle to its diameter
ρ = mass density (nondimensionalized by freestream value)
σ = disturbance state eigenvector
ω = angular frequency of the disturbance state (nondimensionalized by $V_\infty^*/L_{\text{ref}}$ unless otherwise noted)

Subscripts

I = inviscid part
i = imaginary part of complex number
r = real valued part of complex number
V = viscous part
∞ = boundary-layer edge conditions

Superscripts

$\bar{\ }$ = basic state quantity
$\tilde{\ }$ = disturbance state quantity
$*$ = dimensional quantity

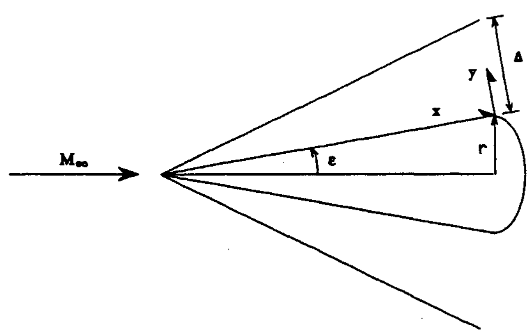

Fig. 1 Coordinate system and definition of geometry.

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Introduction

THE quantitative effect of a cone's transverse surface curvature on the linear stability of a supersonic cone boundary layer has been the subject of controversy. The first to address this issue were Battin and Lin.[1] They considered the transformation of the linear stability equations between Cartesian coordinates and the body-intrinsic coordinate system of a cone (see Fig. 1). They concluded that disturbance state spatial amplification rates are the same in incompressible flows over flat plates and cones with the same edge conditions and the same boundary-layer thicknesses. If this result can also be applied to supersonic compressible flow, then one can deduce (i.e., Mack[2]) from the boundary-layer basic state similarity solutions for flat plates and cones (see White,[3] for example) that the dimensionless amplification rates (nondimensionalized with the Blasius length scale $x^*/\sqrt{Re_x}$) are related by

$$\alpha_{ic} = \sqrt{3}\alpha_{if}$$

where α_{ic} is the spatial amplification rate for the cone flow, and α_{if} is the corresponding amplification rate for flat-plate flow. Again, this relationship holds where the flat plate and cone boundary layers have the same dimensional boundary-layer thicknesses.

Gasperas[4] compared the accuracy of this approximate analysis to that of a linear stability calculation in which he included directly the transverse curvature terms (evaluated locally at each station at which the parallel stability analysis was done). He used conventional definitions of the disturbance state variables (i.e., no geometric terms appeared as factors in their definitions) and concluded that axisymmetric disturbances in a cone boundary layer are actually more stable than predicted by the approximate analysis.

The results presented here are in agreement with this conclusion and in substantial agreement with Gasperas' numerical results for frequencies near that of the second mode. However, the approach taken is somewhat different.

Methodology

The Navier-Stokes equations govern the behavior of the total flow Q: the sum of the basic state $\bar{Q}$ and the infinitesimally small disturbance state $\tilde{Q}$. The basic state is assumed to be steady and laminar and is itself a solution of the Navier-Stokes equations. The disturbance state satisfies a simplified system of equations obtained by linearizing the Navier-Stokes equations about the basic state. These equations are given in the following body-intrinsic coordinate system.

Nondimensionalizing the velocity components by the speed V_∞^* at the boundary-layer edge; all distances by L^* (assumed to be *locally constant*), the density by ρ_∞^*, the temperature by T_∞^*, the pressure and total energy per unit volume by $\rho_\infty^* \mathcal{R}^* T_\infty^*$, and time by V_∞^*/L^*, the equations of conservation of mass, conservation of x momentum, conservation of y momentum, conservation of z momentum, and conservation of energy, in the body-intrinsic coordinate system of Fig. 1 are (i.e., Ref. 5)

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} + H = 0 \tag{1}$$

where

$$Q \equiv h_1 h_3 U \tag{2a}$$

$$U \equiv \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{Bmatrix} \tag{2b}$$

$$E \equiv E_I - E_V \tag{3}$$

$$E_I \equiv h_3 \begin{Bmatrix} \rho u \\ \rho u^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \\ \rho u v \\ \rho u w \\ (E_t + p)u \end{Bmatrix}$$

$$E_V \equiv \frac{h_3}{Re_L} \begin{Bmatrix} 0 \\ S_{xx} \\ S_{yx} \\ S_{zx} \\ \gamma_\infty M_\infty^2 (uS_{xx} + vS_{yx} + wS_{zx}) - q_x \end{Bmatrix}$$

$$F \equiv F_I - F_V \tag{4}$$

$$F_I \equiv h_1 h_3 \begin{Bmatrix} \rho v \\ \rho u v \\ \rho v^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \\ \rho v w \\ (E_t + p)v \end{Bmatrix}$$

$$F_V \equiv \frac{h_1 h_3}{Re_L} \begin{Bmatrix} 0 \\ S_{xy} \\ S_{yy} \\ S_{zy} \\ \gamma_\infty M_\infty^2 (uS_{xy} + vS_{yy} + wS_{zy}) - q_y \end{Bmatrix}$$

$$G \equiv G_I - G_V \tag{5}$$

$$G_I \equiv h_1 \begin{Bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \\ (E_t + p)w \end{Bmatrix}$$

$$G_V \equiv \frac{h_1}{Re_L} \begin{Bmatrix} 0 \\ S_{xz} \\ S_{yz} \\ S_{zz} \\ \gamma_\infty M_\infty^2 (uS_{xz} + vS_{yz} + wS_{zz}) - q_z \end{Bmatrix}$$

$$H \equiv H_I - H_V \tag{6}$$

$$H_I \equiv \begin{Bmatrix} 0 \\ \kappa h_3 \rho u v - m h_1 \left(\rho w^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \right) \sin \epsilon \\ -\kappa h_3 \left(\rho u^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \right) - m h_1 \left(\rho w^2 + \dfrac{p}{\gamma_\infty M_\infty^2} \right) \cos \epsilon \\ m h_1 \rho w (u \sin \epsilon + v \cos \epsilon) \\ 0 \end{Bmatrix}$$

$$H_V \equiv \frac{1}{Re_L} \begin{Bmatrix} 0 \\ \kappa h_3 S_{xy} - m h_1 S_{zz} \sin \epsilon \\ -\kappa h_3 S_{xx} - m h_1 S_{zz} \cos \epsilon \\ m h_1 (S_{xz} \sin \epsilon + S_{yz} \cos \epsilon) \\ 0 \end{Bmatrix}$$

The components of S and q are given in Appendix A.

Disturbance State Variables

If one considers the case $\kappa = 0$ (i.e., flat plates and cones), then $h_1 \equiv 1$. The variable h_2 always equals one, and h_3 is just the local cone radius when evaluated at the body surface. Choosing as disturbance state variables the components of $\tilde{Q}$, the disturbance state mass conservation equation becomes

$$\frac{\partial \tilde{Q}_1}{\partial t} + \frac{\partial \tilde{Q}_2}{\partial x} + \frac{\partial \tilde{Q}_3}{\partial y} + \frac{1}{h_3}\frac{\partial \tilde{Q}_4}{\partial z} = 0 \tag{7a}$$

As Mack[6] has shown, for sufficiently high Mach numbers (~3 for adiabatic flat plates), the most unstable disturbances in supersonic two-dimensional or axisymmetric boundary layers are themselves two dimensional or axisymmetric. For these disturbances, $\partial \tilde{Q}_4/\partial z \equiv 0$. The conservation of mixture mass equation for these boundary layers is thus

$$\frac{\partial \tilde{Q}_1}{\partial t} + \frac{\partial \tilde{Q}_2}{\partial x} + \frac{\partial \tilde{Q}_3}{\partial y} = 0 \tag{7b}$$

The important thing to note is that this equation has the same form regardless of whether the geometry is two dimensional or axisymmetric. (The difference appears in the definition of $\tilde{Q}$.) Moreover, all of the coefficients in the equation are independent of x. The parallel flow approximation is thus *exact* as far as the mass conservation equation is concerned.

The entire set of linear disturbance state equations does not share this feature, however. They can be written in vector form as (for $\kappa = 0$)

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{1}{h_3}\frac{\partial E_I}{\partial U}\frac{\partial \tilde{Q}}{\partial x} + \frac{\partial}{\partial y}\left(\frac{1}{h_3}\frac{\partial F_I}{\partial U}\tilde{Q}\right) + \frac{1}{h_3}\frac{\partial G_I}{\partial U}\frac{\partial \tilde{Q}}{\partial z}$$

$$+ \frac{1}{h_3}\frac{\partial H_I}{\partial U}\tilde{Q} - \frac{\partial \tilde{E}_V}{\partial x} - \frac{\partial \tilde{F}_V}{\partial y} - \frac{\partial \tilde{G}_V}{\partial z} - H_V = 0 \tag{8}$$

(Note that E_I and F_I each have a multiplicative factor of h_3 in their definitions. This factor cancels the corresponding multiplicative factor $1/h_3$ appearing in the linear disturbance state equations.) The inviscid Jacobian matrices

$$\frac{1}{h_3}\frac{\partial E_I}{\partial U}, \quad \frac{1}{h_3}\frac{\partial F_I}{\partial U}, \quad \frac{\partial G_I}{\partial U}, \quad \text{and} \quad \frac{\partial H_I}{\partial U}$$

and the disturbances in the viscous terms are given in Appendix B.

Again, for axisymmetric disturbances, $\partial \tilde{Q}/\partial z \equiv 0$; hence, for these disturbances the fourth term in Eq. (8) vanishes identically. However, there are additional inviscid nonparallel terms that appear in the momentum equations (but not the energy equation) through H_I. However, if one examines the Jacobian matrix

$$\frac{1}{h_3}\frac{\partial H_I}{\partial U}$$

one can see that for sufficiently large x (rather, $\sqrt{Re_x}$), h_3 will be much larger than the boundary-layer thickness (which is the length scale for the disturbance state), and the source term

$$\frac{\partial H_I}{\partial Q}\cdot\tilde{Q}$$

will be negligible. The purpose of this paper is to show that the effect of these terms (as well as the viscous terms like them) on the magnitudes of the spatial amplification rates of a supersonic cone boundary layer is much smaller than

$$\frac{1}{h_3}\frac{\partial h_3}{\partial x}$$

Normal Mode Analysis

Assuming that the basic state $\bar{U}$ satisfies

$$\frac{\partial \bar{U}}{\partial t} \equiv 0 \qquad \text{(steady)}$$

$$\frac{\partial \bar{U}}{\partial z} \equiv 0 \qquad \text{(two dimensional or axisymmetric)}$$

$$\frac{\partial \bar{U}}{\partial x} \cong 0 \qquad \text{(parallel flow)}$$

and assuming that all of the coefficients and source terms in the disturbance state equations are locally constant (i.e., evaluated at the station on the cone or flat plate where the stability analysis is done), normal mode solutions of the form

$$\tilde{Q}(x, y, z, t) = \sigma(y)\exp\left\{i\left[\left[\int_{x_0^*}^{x^*}\alpha^*(\xi^*)\,d\xi^* + \beta z - \omega t\right]\right]\right\} \tag{9}$$

can be found. Here α, β, and ω are in general complex numbers, each of which is also assumed to be locally constant over the characteristic length δ^*, and two of which are explicitly specified. The third number is determined as an eigenvalue of the disturbance state equations (see the following). The integral appears because α^* will vary (even if ω and β are fixed) due to changes in the body geometry and flowfield with respect to x. A dimensional value for α is used to avoid possible confusion arising from the varying length scale $\delta^* = x^*/\sqrt{Re_x}$.

In the *spatial* theory, ω and β are specified real numbers and α is determined as a complex eigenvalue. The *temporal* theory assumes that α and β are specified real numbers and ω is computed as a complex eigenvalue. For flat plates and cones, one must set $\mathrm{Im}(\beta) = \beta_i = 0$ in order for the disturbance state to be bounded in the spanwise direction (or periodic in the azimuthal direction).

Substituting the form (9) for the disturbance state into the linear disturbance state equations yields a set of ordinary differential equations with independent variable y and dependent variables $\sigma(y)$. Since the boundary conditions for the disturbance state are homogeneous, nontrivial solutions exist only for proper values of α, β, and ω. In other words, we have an eigenvalue problem. After discretization, we have an algebraic eigenvalue problem. If α and β are specified, this is a linear algebraic eigenvalue problem because the time derivative in the disturbance state equations appears only to first order. This is the easiest case to consider, for then one can use standard system subroutines that implement the *QZ* algorithm (i.e., Golub and Van Loan[7]) to solve it. The eigenvector can then be found easily using inverse iteration (see, e.g., Malik and Orszag[8] or Stuckert[9]).

Once a charactersitic set of α, β, and ω have been found, a Newton-Raphson iteration can be used to force $\mathrm{Im}(\omega) = 0$. (Typically, only one iteration is needed.) For this, one must compute the group velocity ($\partial\omega/\partial\alpha$, $\partial\omega/\partial\beta$). This was done using the method described by Malik and Orszag.[8] Once one has forced $\mathrm{Im}(\omega) = 0$, the spatial eigenvalue problem has indirectly been solved. Additional characteristic values can then be found by perturbing α (and/or β) and repeating the calculations with inverse iteration.

The important thing to note here concerns the amplification rates for the physical disturbance state $\tilde{U}$. Remember that the components of $\tilde{U}$ and $\tilde{Q}$ are related through their definitions:

$$\tilde{U}_i^* = \frac{\tilde{Q}_i^*}{h_3} \tag{10}$$

(Again, dimensional values are used to avoid confusion due to the variable length scale.) Hence,

$$\frac{\partial \ln|\tilde{U}_i^*|}{\partial x^*} = \frac{\partial}{\partial x^*}\left\{\ln\left(\frac{|\tilde{Q}_i^*|}{h_3^*}\right)\right\} \tag{11a}$$

$$= \frac{\partial \ell n |\tilde{Q}_i^*|}{\partial x^*} - \frac{\partial \ell n (h_3^*)}{\partial x^*} \tag{11b}$$

$$= i\alpha^* - \frac{1}{h_3^*}\frac{\partial h_3^*}{\partial x^*} \tag{11c}$$

For a cone, $h_3^* = x^* \sin \epsilon + y^* \cos \epsilon$, and $\partial h_3^*/\partial x^* = \sin \epsilon$. (Again, at the body surface, h_3^* is just the local cone radius.) Hence, if one assumes that $x^* \sin \epsilon$ is much greater than the boundary-layer thickness, one has $h_3^* \approx x^* \sin \epsilon$ and

$$\frac{1}{h_3^*}\frac{\partial h_3^*}{\partial x^*} \approx \frac{1}{x^*}$$

In terms of the Blasius length scale, $\delta^* = x^*/\sqrt{Re_x}$; this implies

$$\delta^* \frac{\partial \ell n |\tilde{U}_i^*|}{\partial x^*} \approx i\alpha - \frac{1}{\sqrt{Re_x}} \tag{11d}$$

Hence, the physical disturbance state $\tilde{U}$ has the same wave-number vector as that of the scaled disturbance state $\tilde{Q}$, but is more stable at the surface of the cone. However, for any fixed station on the cone,

$$\lim_{y\to\infty}\left(\frac{1}{h_3}\frac{\partial h_3}{\partial x}\right) = 0$$

In this limit the difference between the scaled disturbance state and "physical" disturbance state amplification rates vanishes.

A related statement can be made about the disturbance state amplitude ratio at two different stations x_0 and x_1 on the cone. For any element $\tilde{Q}_i$ of $\tilde{Q}$, this ratio is just

$$\frac{A_Q(x_1)}{A_Q(x_0)} = \exp\left\{-\int_{x_0^*}^{x_1^*} \alpha_i^*(\xi^*)\, d\xi^*\right\} \tag{12a}$$

Again, using the relation between $\tilde{Q}_i$ and $\tilde{U}_i$ established through their definitions, Eq. (12a) implies that the amplitude ratio of U_i is

$$\frac{A_U(x_1^*, y^*)}{A_U(x_0^*, y^*)} = \frac{A_Q(x_1^*)h_3^*(x_0^*, y^*)}{A_Q(x_0^*)h_3^*(x_1^*, y^*)} \tag{12b}$$

The dependence of $A_U(x^*, y^*)$ on y^* is solely due to the dependence of $h_3^*(x^*, y^*)$ on y^*. As discussed earlier, however, if one approximates $h_3^*(x^*, y^*) \approx h_3^*(x^*, 0)$, then the amplitude is a function of x^* only. In this case,

$$\frac{A_U(x_1^*, 0)}{A_U(x_0^*, 0)} = \frac{A_Q(x_1^*)x_0^*}{A_Q(x_0^*)x_1^*} \tag{12c}$$

This can also be expressed in terms of the Reynolds number based on x^*, Re_x:

$$\frac{A_U(x_1^*, 0)}{A_U(x_0^*, 0)} = \frac{A_Q(x_1^*)Re_{x_0}}{A_Q(x_0^*)Re_{x_1}} \tag{12d}$$

Substituting Eq. (12a) for the amplitude ratio of $\tilde{Q}_i$, one obtains

$$\frac{A_U(x_1^*, 0)}{A_U(x_0^*, 0)} = \frac{Re_{x_0}}{Re_{x_1}} \exp\left\{-\int_{x_0^*}^{x_1^*} \alpha_i^*(\xi^*)\, d\xi^*\right\} \tag{12e}$$

In many calculations performed thus far, the planar form of the linear disturbance state equations has been used for boundary-layer flows over cones. However, the ratio of local Reynolds numbers has not been included in the calculation of the amplitude ratios of the disturbances.

Results

The basic state boundary-layer flow over the cone studied here has been approximated by the classical similarity solution (see, e.g., White[3]). This solution was numerically computed at 81 grid points using a shooting method in conjunction with a fourth-order-accurate Runge-Kutta integrator. Although transverse curvature terms were not included in the basic state equations, Mack[10] has shown that this has only a small effect (a couple of percentage points) on the computed spatial amplification rate.

The linear stability analysis was performed using simple second-order-accurate finite differences on the same grid so that no interpolation of the basic state was required. These calculations were performed using a computer program developed for the investigation of chemical nonequilibrium flows. The governing equations for these flows are very involved, and so a relatively simple numerical algorithm was chosen to solve them (see Stuckert[9]). Grid refinement studies for similar flow conditions have indicated that this number of grid points produces spatial growth rates that are at least within the accuracy of Fig. 2. This figure shows the spatial amplification rate as a function of the frequency for the conditions of the experiment of Stetson et al.[11]: a 7-deg cone in a preshock Mach 8.0 flow with total temperature 750 K. The boundary-layer edge Mach number and x Reynolds number are $M_\infty = 6.84$ and $\sqrt{Re_x} = 1730$, respectively.

Three separate curves are presented in Fig. 2 along with Gasperas' numerical results and Stetson's experimental results. These three curves were all computed with constant specific heat ratio $\gamma = 1.4$, the Sutherland viscosity law for air, and Prandtl number $\mu c_p/k = 0.70$. Also, the basic state normal velocity component was taken to be identically zero.

The first curve in Fig. 2 represents the results obtained using the scaled disturbances described earlier with all of the terms like

$$\frac{1}{h_3}\frac{\partial H_I}{\partial U}$$

included but assumed to be locally constant. The second curve shown was obtained by subtracting $1/\sqrt{Re_x}$ from the first curve [the "correction" given in Eq. (11d)].

The third curve is shown to illustrate the importance of the nonparallel body curvature terms in the present formulation of the linear stability equations. It was obtained by using the same cone boundary-layer similarity solution for the basic state, but with $m = 0$ for the linear stability equations (m is the "flow" index used to differentiate between the two-dimensional and axisymmetric cases). Setting $m = 0$ is equivalent to neglecting the surface curvature terms in the governing equations. The disturbance state variables are still $\tilde{Q}$, however, not $\tilde{U}$. Hence, the computed spatial growth rate for $\tilde{U}$ must still be obtained by applying the Reynolds number correction [Eq. (11d)]. The third curve in Fig. 2 represents this corrected spatial growth rate, i.e., the growth rate of the physical disturbances.

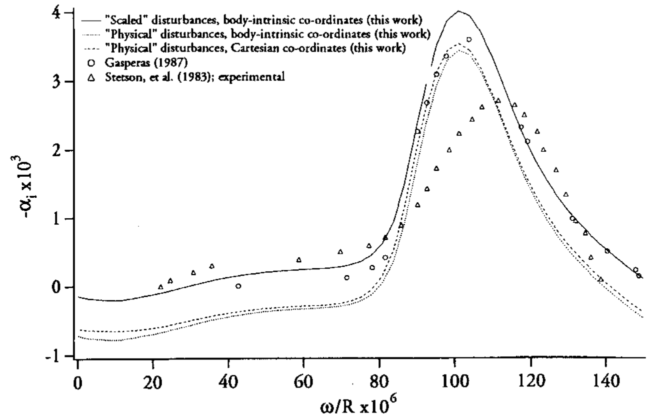

Fig. 2 Adiabatic 7 deg cone ideal gas boundary-layer disturbance state amplification rates at $\sqrt{Re_x} = 1730$, $\beta = 0$ (edge Mach number = 6.84).

One can see from Fig. 2 that the second and third curves are very close together; i.e., the distance between them is much less than $1/\sqrt{Re_x}$. This supports the contention that the nonparallel terms like

$$\frac{1}{h_3}\frac{\partial H_I}{\partial U}$$

that appear in the present formulation of the linear disturbance momentum equations have little effect on the amplification rates of the disturbance state. However, a rigorous proof must depend on a comparison with a truly nonparallel theory.

Comparing the present predicted amplification rates for the physical disturbance state to Gasperas' results, one can see that the present predictions are more stable except at frequencies just below that of the second mode. In fact, except in this neighborhood, the uncorrected growth rates actually agree better with Gasperas' results. Relative to the experimental data, however, all of the theoretical results predict a second mode disturbance that is more unstable and that occurs at a lower frequency.

Conclusions

The use of disturbance state variables scaled by the local radius of curvature for a cone minimizes the nonparallel effects introduced by the nonplanar geometry. With these disturbance state variables, one can compute with reasonable accuracy the spatial growth rates using a locally planar coordinate system. The growth rates of the "physical" disturbances (nondimensionalized by the length scale $x^*/\sqrt{Re_x}$) can then be obtained simply by subtracting $1/\sqrt{Re_x}$. Another simple expression can be found that describes the relationship between the amplitude ratios of $\tilde{U}$ and those of $\tilde{Q}$.

Appendix A: Constitutive Equations

The total energy per unit volume is related to the temperature, density, and kinetic energy of the flow by

$$E_t = \rho c_v T + \tfrac{1}{2}\gamma M_\infty^2 \rho(u^2 + v^2 + w^2) \tag{A1}$$

where $c_v = 1/(\gamma - 1)$ is the specific heat at constant volume.

The pressure is related to the temperature and density of the flow through the ideal gas law:

$$p = \rho T \tag{A2}$$

Assuming that the bulk viscosity of the mixture is zero, the stress tensor components S_{ij} are

$$S_{xx} = \mu(2e_{xx} - \tfrac{2}{3}\nabla \cdot V) \tag{A3}$$

$$S_{xy} = \mu e_{xy} \tag{A4}$$

$$S_{xz} = \mu e_{xz} \tag{A5}$$

$$S_{yz} = \mu e_{yz} \tag{A6}$$

$$S_{yy} = \mu(2e_{yy} - \tfrac{2}{3}\nabla \cdot V) \tag{A7}$$

$$S_{zz} = \mu(2e_{zz} - \tfrac{2}{3}\nabla \cdot V) \tag{A8}$$

where

$$\nabla \cdot V = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial x}(h_3 u) + \frac{\partial}{\partial y}(h_1 h_3 v) + \frac{\partial}{\partial z}(h_1 w)\right]$$

is the dilatation of the fluid, μ is the viscosity, and e is the strain tensor given in the following:

$$e_{xx} = \frac{1}{h_1}\frac{\partial u}{\partial x} + \frac{\kappa v}{h_1} \tag{A9}$$

$$e_{xy} = \frac{1}{h_1}\frac{\partial v}{\partial x} + h_1 \frac{\partial}{\partial y}\left(\frac{u}{h_1}\right) \tag{A10}$$

$$e_{xz} = \frac{h_1}{h_3}\frac{\partial}{\partial z}\left(\frac{u}{h_1}\right) + \frac{h_3}{h_1}\frac{\partial}{\partial x}\left(\frac{w}{h_3}\right) \tag{A11}$$

$$e_{yz} = h_3 \frac{\partial}{\partial y}\left(\frac{w}{h_3}\right) + \frac{1}{h_3}\frac{\partial v}{\partial z} \tag{A12}$$

$$e_{yy} = \frac{\partial v}{\partial y} \tag{A13}$$

$$e_{zz} = \frac{1}{h_3}\frac{\partial w}{\partial z} + \frac{m}{h_3}(u \sin \epsilon + v \cos \epsilon) \tag{A14}$$

Finally, the heat flux components are

$$q_x = -\frac{k}{h_1}\frac{\partial T}{\partial x} \tag{A15}$$

$$q_y = -k\frac{\partial T}{\partial y} \tag{A16}$$

$$q_z = -\frac{k}{h_3}\frac{\partial T}{\partial z} \tag{A17}$$

where k is the thermal conductivity of the gas. For the results shown here, k was computed assuming a constant Prandtl number $\mu c_p/k = 0.70$ and constant specific heat $c_p = 3.5$. The dynamic viscosity was computed using the Sutherland viscosity law for air.

Appendix B: Linear Disturbance State Equations and Jacobian Matrices

It is beneficial to apply a stretching transformation $y = y(\eta)$ for the actual computations. The transformed governing equations can be found from Eq. (8) in the main body of the text by using the chain rule to evaluate the derivatives with respect to y in terms of derivatives with respect to η. Multiplication of the resulting equations by $y_\eta = dy/d\eta$ and use of the disturbance state variables $\tilde{Q} = y_\eta h_3 \tilde{U}$ instead of simply $h_3\tilde{U}$ gives

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{1}{h_3}\frac{\partial E_I}{\partial U}\frac{\partial \tilde{Q}}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{h_3}\frac{\partial F_I}{\partial U}\tilde{Q}\right) + \frac{1}{h_3}\frac{\partial G_I}{\partial U}\frac{\partial \tilde{Q}}{\partial z}$$
$$+\frac{1}{h_3}\frac{\partial H_I}{\partial U}\tilde{Q} - y_\eta \frac{\partial \tilde{E}_V}{\partial x} - \frac{\partial \tilde{F}_V}{\partial \eta} - y_\eta \frac{\partial \tilde{G}_V}{\partial z} - y_\eta \tilde{H}_V = 0 \tag{B1}$$

The Jacobian matrices for E_I, F_I, G_I, and H_I are given here for the case $\kappa \equiv 0$. The disturbances in the viscous terms follow. They are all to be evaluated using basic state quantities.

The ith row in each matrix is the derivative of the ith element of the corresponding inviscid vector with respect to U. For instance, the entry in the ith row and jth column of E_I is the partial derivative of the ith element of E_I with respect to U_j. Only the nonzero elements of these matrices are presented here.

For later reference, it is convenient to first note that

$$\frac{\partial u}{\partial U} = \left(\frac{-u}{\rho}, \frac{1}{\rho}, 0, 0, 0\right) \tag{B2}$$

$$\frac{\partial v}{\partial U} = \left(\frac{-v}{\rho}, 0, \frac{1}{\rho}, 0, 0\right) \tag{B3}$$

$$\frac{\partial w}{\partial U} = \left(\frac{-w}{\rho}, 0, 0, \frac{1}{\rho}, 0\right) \tag{B4}$$

and

$$\frac{1/(\gamma-1)}{\gamma M_\infty^2}\frac{\partial p}{\partial U}=\left(\frac{u^2+v^2+w^2}{2},\ -u,\ -v,\ -w,\ \frac{1}{\gamma M_\infty^2}\right) \tag{B5}$$

$$\frac{\rho/(\gamma-1)}{\gamma M_\infty^2}\frac{\partial T}{\partial U}=\left(\frac{u^2+v^2+w^2}{2}-\frac{T}{(\gamma-1)\gamma M_\infty^2}\right.$$

$$\left.-u,\ -v,\ -w,\ \frac{1}{\gamma M_\infty^2}\right) \tag{B6}$$

$$\frac{1}{h_3}\frac{\partial E_I}{\partial U}$$

$$\frac{\partial E_I}{\partial U}(1,2)=1 \tag{B7}$$

$$\frac{\partial E_I}{\partial U}(2,1)=-u^2+\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho} \tag{B8}$$

$$\frac{\partial E_I}{\partial U}(2,2)=-2u+\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho u} \tag{B9}$$

$$\frac{\partial E_I}{\partial U}(2,3)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho v} \tag{B10}$$

$$\frac{\partial E_I}{\partial U}(2,4)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho w} \tag{B11}$$

$$\frac{\partial E_I}{\partial U}(2,5)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial E_t} \tag{B12}$$

$$\frac{\partial E_I}{\partial U}(3,1)=-uv \tag{B13}$$

$$\frac{\partial E_I}{\partial U}(3,2)=v \tag{B14}$$

$$\frac{\partial E_I}{\partial U}(3,3)=u \tag{B15}$$

$$\frac{\partial E_I}{\partial U}(4,1)=-uw \tag{B16}$$

$$\frac{\partial E_I}{\partial U}(4,2)=w \tag{B17}$$

$$\frac{\partial E_I}{\partial U}(4,4)=u \tag{B18}$$

$$\frac{\partial E_I}{\partial U}(5,1)=u\left(\frac{\partial p}{\partial \rho}-\frac{E_t+p}{\rho}\right) \tag{B19}$$

$$\frac{\partial E_I}{\partial U}(5,2)=u\frac{\partial p}{\partial \rho u}+\frac{E_t+p}{\rho} \tag{B20}$$

$$\frac{\partial E_I}{\partial U}(5,3)=u\frac{\partial p}{\partial \rho v} \tag{B21}$$

$$\frac{\partial E_I}{\partial U}(5,4)=u\frac{\partial p}{\partial \rho w} \tag{B22}$$

$$\frac{\partial E_I}{\partial U}(5,5)=u\left(\frac{\partial p}{\partial E_t}+1\right) \tag{B23}$$

$$\frac{1}{h_3}\frac{\partial F_I}{\partial U}$$

$$\frac{\partial F_I}{\partial U}(1,3)=1 \tag{B24}$$

$$\frac{\partial F_I}{\partial U}(2,1)=-uv \tag{B25}$$

$$\frac{\partial F_I}{\partial U}(2,2)=v \tag{B26}$$

$$\frac{\partial F_I}{\partial U}(2,3)=u \tag{B27}$$

$$\frac{\partial F_I}{\partial U}(3,1)=-v^2+\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho} \tag{B28}$$

$$\frac{\partial F_I}{\partial U}(3,2)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho u} \tag{B29}$$

$$\frac{\partial F_I}{\partial U}(3,3)=-2v+\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho v} \tag{B30}$$

$$\frac{\partial F_I}{\partial U}(3,4)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho w} \tag{B31}$$

$$\frac{\partial F_I}{\partial U}(3,5)=\frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial E_t} \tag{B32}$$

$$\frac{\partial F_I}{\partial U}(4,1)=-vw \tag{B33}$$

$$\frac{\partial F_I}{\partial U}(4,3)=w \tag{B34}$$

$$\frac{\partial F_I}{\partial U}(4,4)=v \tag{B35}$$

$$\frac{\partial F_I}{\partial U}(5,1)=v\left(\frac{\partial p}{\partial \rho}-\frac{E_t+p}{\rho}\right) \tag{B36}$$

$$\frac{\partial F_I}{\partial U}(5,2)=v\frac{\partial p}{\partial \rho u} \tag{B37}$$

$$\frac{\partial F_I}{\partial U}(5,3)=v\frac{\partial p}{\partial \rho v}+\frac{E_t+p}{\rho} \tag{B38}$$

$$\frac{\partial F_I}{\partial U}(5,4)=v\frac{\partial p}{\partial \rho w} \tag{B39}$$

$$\frac{\partial F_I}{\partial U}(5,5)=v\left(\frac{\partial p}{\partial E_t}+1\right) \tag{B40}$$

$$\frac{\partial G_I}{\partial U}$$

$$\frac{\partial G_I}{\partial U}(1,4)=1 \tag{B41}$$

$$\frac{\partial G_I}{\partial U}(2,1)=-uw \tag{B42}$$

$$\frac{\partial G_I}{\partial U}(2,2)=w \tag{B43}$$

$$\frac{\partial G_I}{\partial U}(2,4)=u \tag{B44}$$

$$\frac{\partial G_I}{\partial U}(3,1)=-vw \tag{B45}$$

$$\frac{\partial G_I}{\partial U}(3,3)=w \tag{B46}$$

$$\frac{\partial G_I}{\partial U}(3,4)=v \tag{B47}$$

$$\frac{\partial G_I}{\partial U}(4,1) = -w^2 + \frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho} \tag{B48}$$

$$\frac{\partial G_I}{\partial U}(4,2) = \frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho u} \tag{B49}$$

$$\frac{\partial G_I}{\partial U}(4,3) = \frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho v} \tag{B50}$$

$$\frac{\partial G_I}{\partial U}(4,4) = -2w + \frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho w} \tag{B51}$$

$$\frac{\partial G_I}{\partial U}(4,5) = \frac{1}{\gamma M_\infty^2}\frac{\partial p}{\partial E_t} \tag{B52}$$

$$\frac{\partial G_I}{\partial U}(5,1) = w\left(\frac{\partial p}{\partial \rho} - \frac{E_t + p}{\rho}\right) \tag{B53}$$

$$\frac{\partial G_I}{\partial U}(5,2) = w\frac{\partial p}{\partial \rho u} \tag{B54}$$

$$\frac{\partial G_I}{\partial U}(5,3) = w\frac{\partial p}{\partial \rho v} \tag{B55}$$

$$\frac{\partial G_I}{\partial U}(5,4) = w\frac{\partial p}{\partial \rho w} + \frac{E_t + p}{\rho} \tag{B56}$$

$$\frac{\partial G_I}{\partial U}(5,5) = w\left(\frac{\partial p}{\partial E_t} + 1\right) \tag{B57}$$

$$\frac{\partial H_I}{\partial U}$$

$$\frac{\partial H_I}{\partial U}(2,1) = mw^2 \sin\epsilon - \frac{m \sin\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho} \tag{B58}$$

$$\frac{\partial H_I}{\partial U}(2,2) = -\frac{m \sin\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho u} \tag{B59}$$

$$\frac{\partial H_I}{\partial U}(2,3) = -\frac{m \sin\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho v} \tag{B60}$$

$$\frac{\partial H_I}{\partial U}(2,4) = -2mw \sin\epsilon - \frac{m \sin\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho w} \tag{B61}$$

$$\frac{\partial H_I}{\partial U}(2,5) = -\frac{m \sin\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial E_t} \tag{B62}$$

$$\frac{\partial H_I}{\partial U}(3,1) = mw^2 \cos\epsilon - \frac{m \cos\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho} \tag{B63}$$

$$\frac{\partial H_I}{\partial U}(3,2) = -\frac{m \cos\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho u} \tag{B64}$$

$$\frac{\partial H_I}{\partial U}(3,3) = -\frac{m \cos\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho v} \tag{B65}$$

$$\frac{\partial H_I}{\partial U}(3,4) = -2mw \cos\epsilon - \frac{m \cos\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial \rho w} \tag{B66}$$

$$\frac{\partial H_I}{\partial U}(3,5) = -\frac{m \cos\epsilon}{\gamma M_\infty^2}\frac{\partial p}{\partial E_t} \tag{B67}$$

$$\frac{\partial H_I}{\partial U}(4,1) = -mw(u \sin\epsilon + v \cos\epsilon) \tag{B68}$$

$$\frac{\partial H_I}{\partial U}(4,2) = mw \sin\epsilon \tag{B69}$$

$$\frac{\partial H_I}{\partial U}(4,3) = mw \cos\epsilon \tag{B70}$$

$$\frac{\partial H_I}{\partial U}(4,4) = m(u \sin\epsilon + v \cos\epsilon) \tag{B71}$$

Disturbance State Viscous Jacobians

The derivatives of u, v, w, and T, with respect to U, prevalent in the following equations, are given at the beginning of this appendix.

Definition of Disturbance State Viscous Vectors

The linearized vectors $\tilde{E}_V$, $\tilde{F}_V$, $\tilde{G}_V$, and $\tilde{H}_V$ are

$$y_\eta Re_L \tilde{E}_V = \left\{\begin{matrix} 0 \\ y_\eta h_3 \tilde{S}_{xx} \\ y_\eta h_3 \tilde{S}_{xy} \\ y_\eta h_3 \tilde{S}_{xz} \\ -y_\eta h_3 q_x + y_\eta h_3(\bar{u}\tilde{S}_{xx} + \bar{v}\tilde{S}_{xy} + \bar{w}\tilde{S}_{xz}) \\ +\left(\frac{\partial u}{\partial U}\bar{S}_{xx} + \frac{\partial v}{\partial U}\bar{S}_{xy} + \frac{\partial w}{\partial U}\bar{S}_{xz}\right)\tilde{Q} \end{matrix}\right\}$$

$$y_\eta Re_L \tilde{F}_V = \left\{\begin{matrix} 0 \\ y_\eta h_3 \tilde{S}_{yx} \\ y_\eta h_3 \tilde{S}_{yy} \\ y_\eta h_3 \tilde{S}_{yz} \\ -y_\eta h_3 q_y + y_\eta h_3(\bar{u}\tilde{S}_{yx} + \bar{v}\tilde{S}_{yy} + \bar{w}\tilde{S}_{yz}) \\ +\left(\frac{\partial u}{\partial U}\bar{S}_{yx} + \frac{\partial v}{\partial U}\bar{S}_{yy} + \frac{\partial w}{\partial U}\bar{S}_{yz}\right)\tilde{Q} \end{matrix}\right\}$$

$$y_\eta Re_L \tilde{G}_V = \left\{\begin{matrix} 0 \\ y_\eta h_3 \tilde{S}_{zx} \\ y_\eta h_3 \tilde{S}_{zy} \\ y_\eta h_3 \tilde{S}_{zz} \\ -y_\eta h_3 q_z + y_\eta h_3(\bar{u}\tilde{S}_{zx} + \bar{v}\tilde{S}_{zy} + \bar{w}\tilde{S}_{zz}) \\ +\left(\frac{\partial u}{\partial U}\bar{S}_{zx} + \frac{\partial v}{\partial U}\bar{S}_{zy} + \frac{\partial w}{\partial U}\bar{S}_{zz}\right)\tilde{Q} \end{matrix}\right\}$$

$$y_\eta Re_L \tilde{H}_V = \left\{\begin{matrix} 0 \\ -\frac{m}{h_3} y_\eta h_3 \tilde{S}_{zz} \sin\epsilon \\ -\frac{m}{h_3} y_\eta h_3 \tilde{S}_{zz} \cos\epsilon \\ \frac{m}{h_3}(y_\eta h_3 \tilde{S}_{xz} \sin\epsilon + y_\eta h_3 \tilde{S}_{yz} \cos\epsilon) \\ 0 \end{matrix}\right\}$$

Stress tensor disturbances:

$$y_\eta h_3 \tilde{S}_{xx} = \bar{\mu}\left[y_\eta h_3\left\{\frac{\partial}{\partial Q}\left(2e_{xx} - \frac{2}{3}\nabla\cdot V\right)\right\}\tilde{Q}\right] + \frac{\partial \mu}{\partial U}\tilde{Q}\left(2\bar{e}_{xx} - \frac{2}{3}\nabla\cdot\bar{V}\right)$$

$$y_\eta h_3 \tilde{S}_{xy} = \bar{\mu}\left(y_\eta h_3 \frac{\partial e_{xy}}{\partial Q}\tilde{Q}\right) + \frac{\partial \mu}{\partial U}\tilde{Q}\bar{e}_{xy}$$

$$y_\eta h_3 \tilde{S}_{xz} = \bar{\mu}\left(y_\eta h_3 \frac{\partial e_{xz}}{\partial Q}\tilde{Q}\right) + \frac{\partial \mu}{\partial U}\tilde{Q}\bar{e}_{xz}$$

$$y_\eta h_3 \tilde{S}_{yz} = \bar{\mu}\left(y_\eta h_3 \frac{\partial e_{yz}}{\partial Q}\tilde{Q}\right) + \frac{\partial \mu}{\partial U}\tilde{Q}\bar{e}_{yz}$$

$$y_\eta h_3 \tilde{S}_{yy} = \bar{\mu}\left[y_\eta h_3\left\{\frac{\partial}{\partial Q}\left(2e_{yy} - \frac{2}{3}\nabla\cdot V\right)\right\}\tilde{Q}\right] + \frac{\partial \mu}{\partial U}\tilde{Q}\left(2\bar{e}_{yy} - \frac{2}{3}\nabla\cdot\bar{V}\right)$$

$$y_\eta h_3 \tilde{S}_{zz} = \bar{\mu}\left[y_\eta h_3\left\{\frac{\partial}{\partial Q}\left(2e_{zz} - \frac{2}{3}\nabla\cdot V\right)\right\}\tilde{Q}\right] + \frac{\partial \mu}{\partial U}\tilde{Q}\left(2\bar{e}_{zz} - \frac{2}{3}\nabla\cdot\bar{V}\right)$$

Basic state stress tensor components:

$$\bar{S}_{xx} = \bar{\mu}[2\bar{e}_{xx} - \tfrac{2}{3}\nabla\cdot\bar{V}]$$

$$\bar{S}_{xy} = \bar{\mu}\bar{e}_{xy}$$

$$\bar{S}_{xz} = \bar{\mu}\bar{e}_{xz}$$

$$\bar{S}_{yy} = \bar{\mu}[2\bar{e}_{yy} - \tfrac{2}{3}\nabla\cdot\bar{V}]$$

$$\bar{S}_{yz} = \bar{\mu}\bar{e}_{yz}$$

$$\bar{S}_{zz} = \bar{\mu}[2\bar{e}_{zz} - \tfrac{2}{3}\nabla\cdot\bar{V}]$$

Disturbance state strain rate tensor components:

$$y_\eta h_3 \frac{\partial e_{xx}}{\partial Q}\tilde{Q} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial U}\tilde{Q}\right) - \frac{m\sin\epsilon}{h_3}\frac{\partial u}{\partial U}\tilde{Q}$$

$$y_\eta h_3 \frac{\partial e_{xy}}{\partial Q}\tilde{Q} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial U}\tilde{Q}\right) - \frac{m\sin\epsilon}{h_3}\frac{\partial v}{\partial U}\tilde{Q} + \frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial u}{\partial U}\tilde{Q}\right) - \frac{m\cos\epsilon}{h_3}\frac{\partial u}{\partial U}\tilde{Q}$$

$$y_\eta h_3 \frac{\partial e_{xz}}{\partial Q}\tilde{Q} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial U}\tilde{Q}\right) - \frac{2m\sin\epsilon}{h_3}\frac{\partial w}{\partial U}\tilde{Q} + \frac{1}{h_3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial U}\tilde{Q}\right)$$

$$y_\eta h_3 \frac{\partial e_{yz}}{\partial Q}\tilde{Q} = \frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial w}{\partial U}\tilde{Q}\right) - \frac{2m\cos\epsilon}{h_3}\frac{\partial w}{\partial U}\tilde{Q} + \frac{1}{h_3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial U}\tilde{Q}\right)$$

$$y_\eta h_3 \frac{\partial e_{yy}}{\partial Q}\tilde{Q} = \frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial v}{\partial U}\tilde{Q}\right) - \frac{m\cos\epsilon}{h_3}\frac{\partial v}{\partial U}\tilde{Q}$$

$$y_\eta h_3 \frac{\partial e_{zz}}{\partial Q}\tilde{Q} = \frac{1}{h_3}\frac{\partial w}{\partial U}\frac{\partial\tilde{Q}}{\partial z} + \frac{m}{h_3}\left(\frac{\partial u}{\partial U}\sin\epsilon + \frac{\partial v}{\partial U}\cos\epsilon\right)\tilde{Q}$$

Finally, the disturbance state dilatation is

$$y_\eta h_3 \frac{\partial(\nabla\cdot V)}{\partial Q}\tilde{Q} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial U}\tilde{Q}\right) + \frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial v}{\partial U}\tilde{Q}\right) + \frac{1}{h_3}\frac{\partial w}{\partial U}\frac{\partial\tilde{Q}}{\partial z}$$

Basic State Strain Rate Tensor Components

The basic state strain rate tensor is also needed and is given in the following, subject to the approximation that $\partial\bar{U}/\partial x \approx 0$ and $\partial\bar{U}/\partial x \equiv 0$:

$$\bar{e}_{xx} = 0$$

$$\bar{e}_{xy} = \eta_y\frac{\partial\bar{u}}{\partial\eta}$$

$$\bar{e}_{xz} = -\frac{m\bar{w}\sin\epsilon}{h_3}$$

$$\bar{e}_{yz} = \eta_y\frac{\partial\bar{w}}{\partial\eta} - \frac{m\bar{w}\cos\epsilon}{h_3}$$

$$\bar{e}_{yy} = \eta_y\frac{\partial\bar{v}}{\partial\eta}$$

$$\bar{e}_{zz} = \frac{m}{h_3}(\bar{u}\sin\epsilon + \bar{v}\cos\epsilon)$$

The basic state dilatation is

$$\nabla\cdot\bar{V} = \eta_y\frac{\partial\bar{w}}{\partial\eta} + \frac{m}{h_3}(u\sin\epsilon + \bar{v}\cos\epsilon) - \frac{m\bar{w}\cos\epsilon}{h_3}$$

Disturbance state heat/energy flux components:

$$y_\eta h_3\tilde{q}_x = -\bar{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial U}\tilde{Q}\right) - \frac{m\sin\epsilon}{h_3}\frac{\partial T}{\partial U}\tilde{Q}\right\}$$

$$y_\eta h_3\tilde{q}_y = -\bar{k}\frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial T}{\partial U}\tilde{Q}\right) + \frac{m\bar{k}\cos\epsilon}{h_3}\frac{\partial T}{\partial U}\tilde{Q} - \eta_y\frac{\partial k}{\partial U}\tilde{Q}\frac{\partial\bar{T}}{\partial\eta}$$

$$y_\eta h_3\tilde{q}_z = -\frac{\bar{k}}{h_3}\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial U}\tilde{Q}\right)$$

Basic state disturbance heat/energy flux components:

$$\bar{q}_x = 0$$

$$\bar{q}_y = -\bar{k}\eta_y\frac{\partial\bar{T}}{\partial\eta}$$

$$\bar{q}_z = 0$$

Derivative of the Vector $\tilde{E}_V$ with Respect to x

The derivative of the viscous vector $\tilde{E}_V$ with respect to x is

$$Re_L\frac{\partial\tilde{E}_V}{\partial x} = \begin{Bmatrix} 0 \\ \frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xx}) \\ \frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xy}) \\ \frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xz}) \\ \bar{u}\frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xx}) + \bar{v}\frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xy}) \\ +\bar{w}\frac{\partial}{\partial x}(y_\eta h_3\tilde{S}_{xz}) - \frac{\partial}{\partial x}(y_\eta h_3\tilde{q}_x) \\ +\left(\frac{\partial u}{\partial U}\bar{S}_{xx} + \frac{\partial v}{\partial U}\bar{S}_{xy} + \frac{\partial w}{\partial U}\bar{S}_{xz}\right)\frac{\partial\tilde{Q}}{\partial x} \\ +\left(\frac{\partial u}{\partial U}\frac{\partial\tilde{S}_{xx}}{\partial x} + \frac{\partial v}{\partial U}\frac{\partial\tilde{S}_{xy}}{\partial x} + \frac{\partial w}{\partial U}\frac{\partial\tilde{S}_{xz}}{\partial x}\right)\tilde{Q} \end{Bmatrix}$$

To evaluate the individual terms in this vector, make the approximation that $\partial\bar{U}/\partial x \approx 0$ and $\partial\bar{U}/\partial z \equiv 0$. (This approximation is made throughout this appendix unless explicitly indicated otherwise.) The components of the vector $\partial\tilde{E}_V/\partial x$ can then be easily determined from the following.

Disturbance State Stress Components

The derivative with respect to x of the first column of the disturbance state stress tensor is

$$\frac{\partial}{\partial x}(y_\eta h_3 \tilde{S}_{xx}) = \bar{\mu}\frac{\partial}{\partial x}\left[y_\eta h_3\left\{\frac{\partial}{\partial Q}\left(2e_{xx} - \frac{2}{3}\nabla\cdot V\right)\right\}\tilde{Q}\right] + \frac{\partial\mu}{\partial U}\frac{\partial\tilde{Q}}{\partial x}\left(2\bar{e}_{xx} - \frac{2}{3}\nabla\cdot\bar{V}\right) + \frac{\partial\mu}{\partial U}\tilde{Q}\frac{\partial}{\partial x}\left(2\bar{e}_{xx} - \frac{2}{3}\nabla\cdot\bar{V}\right)$$

$$\frac{\partial}{\partial x}(y_\eta h_3 \tilde{S}_{xy}) = \bar{\mu}\frac{\partial}{\partial x}\left(y_\eta h_3\frac{\partial e_{xy}}{\partial Q}\tilde{Q}\right) + \frac{\partial\mu}{\partial U}\frac{\partial\tilde{Q}}{\partial x}\bar{e}_{xy} + \frac{\partial\mu}{\partial U}\tilde{Q}\frac{\partial\bar{e}_{xy}}{\partial x}$$

$$\frac{\partial}{\partial x}(y_\eta h_3 \tilde{S}_{xz}) = \bar{\mu}\frac{\partial}{\partial x}\left(y_\eta h_3\frac{\partial e_{xz}}{\partial Q}\tilde{Q}\right) + \frac{\partial\mu}{\partial U}\frac{\partial\tilde{Q}}{\partial x}\bar{e}_{xz} + \frac{\partial\mu}{\partial U}\tilde{Q}\frac{\partial\bar{e}_{xz}}{\partial x}$$

Basic State Stress Components

The derivative with respect to x of the first column of the basic state stress tensor is

$$\frac{\partial\bar{S}_{xx}}{\partial x} = \bar{\mu}\frac{\partial}{\partial x}\left[2\bar{e}_{xx} - \frac{2}{3}\nabla\cdot\bar{V}\right]$$

$$\frac{\partial\bar{S}_{xy}}{\partial x} = \bar{\mu}\frac{\partial\bar{e}_{xy}}{\partial x}$$

$$\frac{\partial\bar{S}_{xz}}{\partial x} = \bar{\mu}\frac{\partial\bar{e}_{xz}}{\partial x}$$

$$\frac{\partial\bar{S}_{yy}}{\partial x} = \bar{\mu}\frac{\partial}{\partial x}\left[2\bar{e}_{yy} - \frac{2}{3}\nabla\cdot\bar{V}\right]$$

$$\frac{\partial\bar{S}_{yz}}{\partial x} = \bar{\mu}\frac{\partial\bar{e}_{yz}}{\partial x}$$

$$\frac{\partial\bar{S}_{zz}}{\partial x} = \bar{\mu}\frac{\partial}{\partial x}\left[2\bar{e}_{zz} - \frac{2}{3}\nabla\cdot\bar{V}\right]$$

Disturbance state strain rate components:

$$\frac{\partial}{\partial x}\left(y_\eta h_3\frac{\partial e_{xx}}{\partial Q}\tilde{Q}\right) = \frac{\partial u}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x^2} + \frac{m\sin\epsilon}{h_3}\frac{\partial u}{\partial U}\left\{-\frac{\partial\tilde{Q}}{\partial x} + \frac{\sin\epsilon}{h_3}\tilde{Q}\right\}$$

$$\frac{\partial}{\partial x}\left(y_\eta h_3\frac{\partial e_{xy}}{\partial Q}\tilde{Q}\right) = \frac{\partial v}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x^2} + \frac{m}{h_3}\left(\frac{\partial u}{\partial U}\cos\epsilon + \frac{\partial v}{\partial U}\sin\epsilon\right) \times\left(-\frac{\partial\tilde{Q}}{\partial x} + \frac{\sin\epsilon}{h_3}\tilde{Q}\right) + \frac{\partial}{\partial\eta}\left\{\left(\eta_y\frac{\partial u}{\partial U}\right)\frac{\partial\tilde{Q}}{\partial x}\right\}$$

$$\frac{\partial}{\partial x}\left(y_\eta h_3\frac{\partial e_{xz}}{\partial Q}\tilde{Q}\right) = \frac{\partial w}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x^2} + \frac{2m\sin\epsilon}{h_3}\frac{\partial w}{\partial U}\left(-\frac{\partial\tilde{Q}}{\partial x} + \frac{\sin\epsilon}{h_3}\tilde{Q}\right) + \frac{1}{h_3}\frac{\partial u}{\partial U}\left[\frac{\partial}{\partial z}\frac{\partial\tilde{Q}}{\partial x} - \frac{m\sin\epsilon}{h_3}\frac{\partial\tilde{Q}}{\partial z}\right]$$

$$\frac{\partial}{\partial x}\left[y_\eta h_3\frac{\partial(\nabla\cdot V)}{\partial Q}\tilde{Q}\right] = \frac{\partial u}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x^2} + \frac{1}{h_3}\frac{\partial w}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x\partial z} + \frac{\partial}{\partial\eta}\left(\eta_y\frac{\partial v}{\partial U}\frac{\partial\tilde{Q}}{\partial x}\right) - \frac{m\sin\epsilon}{h_3^2}\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial U}\tilde{Q}\right)$$

Basic State Strain Rate Components

The derivative with respect to x of the first column of the basic state strain rate tensor is

$$\frac{\partial\bar{e}_{xx}}{\partial x} = 0$$

$$\frac{\partial\bar{e}_{xy}}{\partial x} = 0$$

$$\frac{\partial\bar{e}_{xz}}{\partial x} = m\bar{w}\left(\frac{\sin\epsilon}{h_3}\right)^2$$

$$\frac{\partial(\nabla\cdot\bar{V})}{\partial x} = -m\bar{u}\left(\frac{\sin\epsilon}{h_3}\right)^2$$

Disturbance State Heat/Energy Flux Components

The derivative with respect to x of the disturbance in the heat flux gradient in the x direction is

$$\frac{\partial}{\partial x}(y_\eta h_3 q_x) = -\bar{k}\frac{\partial T}{\partial U}\frac{\partial^2\tilde{Q}}{\partial x^2} + m\frac{\bar{k}\sin\epsilon}{h_3}\frac{\partial T}{\partial U}\left(-\frac{\partial\tilde{Q}}{\partial x} + \frac{\sin\epsilon}{h_3}\tilde{Q}\right)$$

Basic State Disturbance Heat/Energy Flux Components

The derivative with respect to x of the basic state heat flux gradient in the x direction is

$$\frac{\partial\bar{q}_x}{\partial x} = 0$$

Summary

Having identified the viscous terms of the linear disturbance equations, one can now easily obtain the corresponding terms in the normal mode equations. This is done simply by replacing $\partial^{m+n}\tilde{Q}_i/\partial x^m\partial x^n$ with $(-i\alpha)^m\,(-i\beta)^n\tilde{Q}$, where $i \equiv \sqrt{-1}$. The results of this straightforward procedure are not presented here.

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